

Quantum Kalb-Ramond Field in D-dimensional de Sitter Spacetimes

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Abstract

In this work we investigate the quantum theory of the Kalb-Ramond fields propagating in D -dimensional de Sitter spacetimes using the dynamic invariant method developed by Lewis and Riesenfeld [J. Math. Phys. 10, 1458 (1969)] to obtain the solution of the time-dependent Schrödinger equation. The wave function is written in terms of a c -number quantity satisfying of the Milne-Pinney equation, whose solution can be expressed in terms of the two independent solutions of the respective equation of motion. We obtain the exact solution for the quantum Kalb-Ramond field in the de Sitter background and discuss its relation with the Cremmer-Scherk-Kalb-Ramond model.

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1 Introduction

Although the Quantum Field Theory in the Minkowski space is well developed, there is still a question related to the behavior of fields in different backgrounds in cosmological scales. The string theory is likely the best tool used to solve the problem of quantum gravity [2, 3, 4], even though other problems can be analyzed in the light of fields propagating in curved spacetimes [5]. Several interesting results were found in time-dependent backgrounds, namely: (i) black hole evaporation [6], (ii) the Unruh [7] and Casimir [8] effects and (iii) the particle creation [9, 10, 11].

The quantum effects of a massive scalar field in the de Sitter spacetime from a Schrödinger-picture point of view has been investigated in Ref. [12]. The light propagation through time-dependent dielectric linear media in the absence of free charges and in a curved spacetime from a classical and a quantum point of view was presented in Ref. [13]. Scalar fields have also been investigated in Ref. [14], where the coherent and squeezed states were investigated.

In these previous works (and in Refs. [15, 16]), the authors used the Lewis and Riesenfeld method [1] to study the quantization of fields propagating in $D = 4$. Recently, we also have used the Lewis and Riesenfeld method to study the quantization of both scalar [17] and electromagnetic fields [18] in arbitrary D -dimensions. In Ref. [17] we found that the Bunch-Davies thermal bath depends on the choice of D and the conformal parameter ξ , which is important in extra-dimension physics, e.g. in the Randall-Sundrum models. In Ref. [18] we found, as expected, that only for $D = 4$ there is no thermal bath.

In this paper we study the problem of quantizing the Kalb-Ramond field in the D -dimensional de Sitter spacetime. The Kalb-Ramond field has been studied in $D = 4$ as a description of an axion field [19, 20]. In extra dimensional scenarios, it has been used to describe torsion and brane interactions [21, 22]. Although the concept of extra dimension has been introduced long time ago, it has now received great attention due to the String Theory and the Randall-Sundrum models [23, 24]. The effective action for the standard model and the localization of fields within of an extra dimensional scenario have been studied in Refs. [25] and [26, 27], respectively. The Kalb-Ramond field emerges naturally in the spectrum of the closed string [2, 3] and its presence may induce optical activity in the visible brane [28].

This paper is organized as follows. In Sec. II the correspondence between the Kalb-Ramond field placed in a de Sitter background and a time-dependent harmonic oscillator is obtained. In Sec. III, we use the Lewis and Riesenfeld [1] method to obtain the solutions of the related Schrödinger equation in arbitrary D . Section IV summarizes the results.

2 Decomposition of the Kalb-Ramond Field

The action for the Kalb-Ramond field in the de Sitter spacetime is given by

$$S = - \int d^D x \sqrt{-g} g^{\mu\nu} g^{\alpha\beta} g^{\gamma\theta} H_{\mu\alpha\gamma} H_{\nu\beta\theta}, \quad (1)$$

where $g_{\mu\nu} = (-1, e^{2H_0 t}, e^{2H_0 t}, e^{2H_0 t})$ is the metric and $H_{\mu\nu\alpha} = \partial_{[\mu} B_{\nu\alpha]}$. Differently from the procedure in [17] we will perform the field decomposition directly in the equations of motion. From the above action we obtain the equation of motion

$$\partial_\nu (\sqrt{-g} g^{\mu\nu} g^{\alpha\beta} g^{\gamma\theta} H_{\mu\alpha\gamma}) = 0. \quad (2)$$

Due to the gauge invariance we can impose the conditions $B_{i0} = 0$, where $i = 1, 2, \dots, D$, and due to the transversality condition we have $\partial^i B_{ij} = 0$. Therefore, the only non trivial equations are

$$\ddot{B}_{ij} + (D-5) \frac{\dot{a}}{a} \dot{B}_{ij} - \frac{1}{a^2} \nabla^2 B_{ij} = 0. \quad (3)$$

Now we decompose the field as

$$B_{ij}(\vec{x}, t) = \sum_\lambda \int \frac{d^{D-1}k}{(2\pi)^{D-1}} \epsilon_{ij}^\lambda(u_1^{(\lambda)}(t) e^{i\vec{k} \cdot \vec{x}} + u_2^{(\lambda)}(t) e^{-i\vec{k} \cdot \vec{x}}), \quad (4)$$

where ϵ_{ij} represents the polarization that obey the Gauge condition $k_i \epsilon_{ij} = 0$ and λ run from 1 to $(D-2)(D-3)/2$. Using Eqs. (3) and (4) we get the equation of motion for the modes u given by

$$\ddot{u}_\sigma + (D-5) \frac{\dot{a}}{a} \dot{u}_\sigma + \frac{1}{a^2} k^2 u_\sigma = 0, \quad (5)$$

where σ represents all the index of u , namely u_1^λ or u_2^λ .

Consider now the Hamiltonian of the harmonic oscillator with time-dependent mass and frequency given by

$$H(t) = \frac{p^2}{2m(t)} + \frac{1}{2} m(t) \omega^2(t) q^2. \quad (6)$$

The equation of motion is given by

$$\ddot{q} + \frac{\dot{m}}{m} \dot{q} + \omega^2 q = 0, \quad (7)$$

which is very similar to equation (5). Therefore our system can be considered as a time-dependent harmonic oscillator if we use $m(t) = a^{(D-5)}$ and $\omega = k a^{-1}$. A very powerful tool to perform the quantization of this system is discussed this in the next section.

3 Quantization of the Kalb-Ramond Field in the de Sitter Spacetime

Consider a time-dependent harmonic oscillator described by Eq. (6). It is well known that an invariant for Eq. (6) is given by [29]

$$I = \frac{1}{2} \left[\left(\frac{q}{\rho} \right)^2 + (\rho p - m \dot{\rho} q)^2 \right], \quad (8)$$

where $\rho(t)$ satisfies the generalized Milne-Pinney [30, 31] (MP) equation

$$\ddot{\rho} + \gamma(t)\dot{\rho} + \omega^2(t)\rho = \frac{1}{m^2(t)\rho^3}, \quad (9)$$

and $\gamma(t) = \dot{m}(t)/m(t)$. The invariant $I(t)$ satisfies the equation

$$\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{1}{i\hbar} [I, H] = 0, \quad (10)$$

and can be considered hermitian if we choose only the real solutions of Eq. (9). Its eigenfunctions, $\phi_n(q, t)$, are assumed to form a complete orthonormal set with time-independent discrete eigenvalues, $\lambda_n = (n + \frac{1}{2})\hbar$.

Consider the time-dependent creation ($a^\dagger(t)$) and annihilation ($a(t)$) operators defined as

$$a^\dagger(t) = \left(\frac{1}{2\hbar} \right)^{1/2} \left[\left(\frac{q}{\rho} - i(\rho p - m(t)\dot{\rho}q) \right) \right]; \quad (11)$$

$$a(t) = \left(\frac{1}{2\hbar} \right)^{1/2} \left[\left(\frac{q}{\rho} + i(\rho p - m(t)\dot{\rho}q) \right) \right], \quad (12)$$

where $[a^\dagger(t), a(t)] = 1$. In terms of $a(t)$ and $a^\dagger(t)$ the invariant I (see Eq. 8) can be written as

$$I = \hbar \left(a^\dagger(t)a(t) + \frac{1}{2} \right). \quad (13)$$

Let $|n, t\rangle$ be the eigenstates of I . Therefore the following relations hold

$$a(t)|n, t\rangle = \sqrt{n}|n-1, t\rangle; \quad (14)$$

$$a^\dagger(t)|n, t\rangle = \sqrt{n+1}|n+1, t\rangle; \quad (15)$$

$$I|n, t\rangle = \left(n + \frac{1}{2} \right) \hbar |n, t\rangle. \quad (16)$$

The exact solution of the Schrödinger equation for the time-dependent harmonic oscillator reads

$$|\psi_n\rangle = e^{i\theta_n(t)} |n, t\rangle, \quad (17)$$

where the phase functions $\theta_n(t)$ satisfy the equation

$$\hbar \frac{d\theta_n(t)}{dt} = \left\langle \phi_n(q, t) \left| i\hbar \frac{\partial}{\partial t} - H(t) \right| \phi_n(q, t) \right\rangle. \quad (18)$$

In the coordinate space $\psi_n(q, t)$ reads

$$\begin{aligned} \psi_n(q, t) = & e^{i\theta_n(t)} \left(\frac{1}{\pi^{1/2} \hbar^{1/2} n! 2^n \rho} \right)^{1/2} \times \\ & \exp \left\{ \frac{im(t)}{2\hbar} \left[\frac{\dot{\rho}}{2\hbar} + \frac{i}{m(t)\rho^2(t)} \right] q^2 \right\} \times \\ & H_n \left(\frac{1}{\sqrt{\hbar}} \frac{q}{\rho} \right), \end{aligned} \quad (19)$$

where H_n is the Hermite polynomial of order n . For a given $m(t)$ and $\omega(t)$ one has to solve Eq. (9) to obtain the exact solutions to the time-dependent Schrödinger equation for $H(t)$ given by Eq. (6). With $m(t) = a^{D-5}(t)$ and $\omega(t) = ka^{-1}(t)$ the MP equation reads

$$\ddot{\rho} + (D-5) \frac{\dot{a}}{a} \dot{\rho} + k^2 a^{-2}(t) \rho = \frac{a^{-2(D-5)}(t)}{\rho^3}.. \quad (20)$$

According to Ref. [32] the solution of the MP equation is related to the solutions of Eq. (5). By setting $dt = a(t)d\eta$ and $u_\sigma = \Omega \bar{u}_\sigma$, we rewrite Eq. (5) as

$$\begin{aligned} \bar{u}_\sigma'' + \left(2a \frac{\dot{\Omega}}{\Omega} - \dot{a} + (D-5)\dot{a} \right) \bar{u}_\sigma' \\ + \left(k^2 + a^2 \frac{\ddot{\Omega}}{\Omega} + (D-5)a\dot{a} \frac{\dot{\Omega}}{\Omega} \right) \bar{u}_\sigma = 0, \end{aligned} \quad (21)$$

where the prime and the dot denote the differentiation with respect to η and t respectively. By choosing $\Omega = a^{-\frac{D-5}{2}}$ we find

$$\begin{aligned} \bar{u}_\sigma'' - \dot{a} \bar{u}_\sigma' + \left[k^2 - \frac{(D-5)}{2} a \ddot{a} \right. \\ \left. + \frac{(D-5)(D-7)}{4} \dot{a}^2 \right] \bar{u}_\sigma = 0. \end{aligned} \quad (22)$$

Next, let us consider the de Sitter spacetime where $a = e^{Ht}$, and with $dt = a(t)d\eta$ we get the relations

$$\eta = -\frac{e^{-Ht}}{H} = \frac{1}{Ha(t)}, \quad \dot{a} = -\frac{1}{\eta}, \quad \ddot{a} = -\frac{H}{\eta}, \quad (23)$$

to obtain

$$\bar{u}_\sigma'' + \frac{1}{\eta} \bar{u}_\sigma' + \left(k^2 - \frac{(D-5)^2}{4\eta^2} \right) \bar{u}_\sigma = 0, \quad (24)$$

or

$$\left[\frac{d^2}{d(k\eta)^2} + \frac{1}{(k\eta)} \frac{d}{d(k\eta)} + \left(1 - \frac{\nu^2}{(k\eta)^2} \right) \right] \bar{u}_\sigma = 0, \quad (25)$$

where $\nu^2 = \frac{(D-5)^2}{4}$. This is a Bessel equation whose two independent solutions are $J_\nu(k|\eta|)$ and $N_\nu(k|\eta|)$. In terms of u_σ , we find

$$\begin{cases} (a^{\frac{-(D-5)}{2}} J_\nu(k|\eta|) \\ a^{\frac{-(D-5)}{2}} N_\nu(k|\eta|) \end{cases},$$

and the solution of the MP equation reads

$$\rho = \frac{a^{\frac{-(D-5)}{2}} H 2^{\frac{1}{2}}}{\pi} \left[A J_\nu^2 + B N_\nu^2 + \left(AB - \frac{\pi^2}{4H^2} \right)^{\frac{1}{2}} J_\nu N_\nu \right]^{\frac{1}{2}}, \quad (26)$$

where A and B are real constants. The values of these constants are related to the choice of the vacuum. By choosing the Bunch-Davies vacuum, which is the adiabatic vacuum at early times we set $A = B = \pi/2H$ and ρ reads, for $a = 1/H\eta$

$$\rho = (|H|\eta)^{\frac{(D-5)}{2}} \sqrt{\frac{H}{\pi}} [J_\nu^2 + N_\nu^2]^{\frac{1}{2}}. \quad (27)$$

Since the problem of the KR field quantization in the de Sitter background was reduced to solve the Schrödinger equation for the harmonic oscillator with time dependent mass and frequency, the quantization procedure ends by substituting Eq. (27) into Eq. (19).

4 Concluding Remarks

In this paper we used the Lewis and Riesenfeld method to obtain the time-dependent Schrödinger states emerging from the quantization of the Kalb-Ramond field in the D -dimensional de Sitter spacetime. It is well-known that a challenge in obtaining the exact solution (see Eq. (19)) for the SE with H given in Eq. (6), is to solve the Milne-Pinney (in terms of the universal scale factor of universe $a(t)$) equation (21).

In a previous article [17] the authors has found that for $D = 4$ the solution for the electromagnetic field is time independent. This is related to the fact the in four dimension the Electromagnetic field is conformal and therefore comoving referential do not feel a Bunch-Davies thermal bath. In the case of the KR field, the solution (27) show us that for $D = 6$ we have a time independent solution, i.e. the KR field is conformal only in $D = 6$. Therefore a solution with arbitrary $a(t)$ can be found trivially in the conformal time. However if we consider the space with dimensions $D \neq 6$, the KR field loses its conformality. Therefore this may have important consequences depending on the model considered.

For $D \neq 6$ we can easily verify that $\rho = \text{constant}$ is not a solution of the equation above. In the case of a de Sitter spacetime, we found the general solution and as we expected, the KR field has more interesting solutions. For $D \neq 6$ the solution are time-dependent, what give us particle production and the comoving referential in this spaces should feel a thermal bath. The drawback in

this analysis is that it is very difficult to imagine a way to measure the associated temperature of the Kalb-Ramond field. However, in interacting models where the KR field is coupled to the EM field, some measurable effects may arise. An example of this is the Cremmer-Scherk-Kalb-Ramond [34] and therefore, at least in principle, we could have some correction to the Cosmic Microwave Background.

At last, we would like to point out that the procedure described may be used to trace the present properties of the quantum KR field back to the recombination era in an arbitrary D -dimensional universe. This would be a much more interesting phenomenological result and is under analysis by the authors.

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References

- [1] H. R. Lewis and W. B. Riesenfeld, J. Math. Phys. **10**, 1458 (1969).
- [2] J. Polchinski, “String theory. Vol. 1: An introduction to the bosonic string,” *Cambridge, UK: Univ. Pr. (1998) 402 p*
- [3] J. Polchinski, “String theory. Vol. 2: Superstring theory and beyond,” *Cambridge, UK: Univ. Pr. (1998) 531 p*
- [4] N. Berkovits, JHEP **0004**, 018 (2000) [arXiv:hep-th/0001035].
- [5] N. D. Birrell and P. C. W. Davies, “QUANTUM FIELDS IN CURVED SPACE,” *Cambridge, UK: Univ. Pr. (1982) 340p*
- [6] S. W. Hawking, Nature **248**, 30 (1974).
- [7] L. C. B. Crispino, A. Higuchi and G. E. A. Matsas, Rev. Mod. Phys. **80**, 787 (2008) [arXiv:0710.5373 [gr-qc]].
- [8] A. A. Saharian and T. A. Vardanyan, Class. Quant. Grav. **26**, 195004 (2009) [arXiv:0907.1149 [hep-th]].
- [9] L. Parker, Phys. Rev. Lett. **21**, 562 (1968).
- [10] L. Parker, Phys. Rev. **183**, 1057 (1969).
- [11] L. Parker, Phys. Rev. D **3**, 346 (1971) [Erratum-ibid. D **3**, 2546 (1971)].
- [12] C. E. F. Lopes, I. A. Pedrosa, C. Furtado and A. M. de M. Carvalho, J. Math. Phys., **50**, 083511 (2009).

- [13] I. A. Pedrosa, Claudio Furtado and Alexandre Rosas, *Europhys. Lett.* **94**, 30002 (2011).
- [14] A. M. de M. Carvalho, C. Furtado, I. A. Pedrosa, *Physical Review D* **70**, 123523 (2004).
- [15] I. A. Pedrosa, *Phys. Rev. A* **55**, 3219 (1997).
- [16] I. A. Pedrosa, A. Rosas and I. Guedes, *J. Phys Gen.* **38**, 7757 (2005).
- [17] G. Alencar, I. Guedes, R. R. Landim and R. N. C. Filho, *Europhys. Lett.* **98**, 11001 (2012) [arXiv:1110.0481 [hep-th]].
- [18] G. Alencar, I. Guedes, R. R. Landim and R. N. C. Filho, arXiv:1107.2558 [hep-th].
- [19] M. Gasperini and G. Veneziano, *Phys. Rev. D* **59**, 043503 (1999) [hep-ph/9806327].
- [20] M. Giovannini, *Phys. Rev. D* **59**, 063503 (1999) [hep-th/9809185].
- [21] B. Mukhopadhyaya, S. Sen and S. SenGupta, *Phys. Rev. D* **79**, 124029 (2009) [arXiv:0903.0722 [hep-th]].
- [22] F. A. Barone, F. E. Barone and J. A. Helayel-Neto, *Phys. Rev. D* **84**, 065026 (2011) [arXiv:1009.3692 [hep-th]].
- [23] L. Randall and R. Sundrum, “An alternative to compactification,” *Phys. Rev. Lett.* **83**, 4690 (1999) [arXiv:hep-th/9906064].
- [24] L. Randall and R. Sundrum, “A large mass hierarchy from a small extra dimension,” *Phys. Rev. Lett.* **83**, 3370 (1999) [arXiv:hep-ph/9905221].
- [25] C. Csaki, J. Erlich and J. Terning, *Phys. Rev. D* **66**, 064021 (2002) [hep-ph/0203034].
- [26] R. R. Landim, G. Alencar, M. O. Tahim and R. N. C. Filho, *JHEP* **1108**, 071 (2011) [arXiv:1105.5573 [hep-th]].
- [27] R. R. Landim, G. Alencar, M. O. Tahim and R. N. Costa Filho, *JHEP* **1202**, 073 (2012) [arXiv:1110.5855 [hep-th]].
- [28] S. Kar, S. SenGupta and S. Sur, *Phys. Rev. D* **67**, 044005 (2003) [hep-th/0210176].
- [29] J. F. Cariñena, J. de Lucas, *Int. J. Geom. Methods Mod. Phys.*, **6**:4 (2009), 683–699.
- [30] E. W. Milne, *Phys. Rev.* **35** (1930) 863.
- [31] E. Pinney, *Proc. Am. Math. Soc.* **1** (1950) 681.

- [32] G. E. Prince and C. J. Eliezer, J. Phys. A: Math. Gen. **13**, 815(1980).
- [33] F. Finelli, A. Gruppuso and G. Venturi, Class. Quant. Grav. **16**, 3923 (1999) [arXiv:gr-qc/9909007].
- [34] E. Cremmer and J. Scherk, Nucl. Phys. B **72**, 117 (1974).